Efficiency of Sum Constructed Automorphic Symmetric Balanced Incomplete Block Designs

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Abstract
Several construction methods have been introduced to build the elements of balanced incomplete block designs (BIBDs) for specific parameters, with different techniques suggested for testing their existence. There is still no general technique to determine the existence of the designs that have been realized. In this study the efficiencies of some given automorphic symmetric balanced incomplete block designs (AUSBIBD) formed by sum construction method have been presented in details alongside existence and non-existence of such designs. The process involved the application of sum construction method to give new designs of parameters D (v, b, $\lambda_1+\lambda_2$) and an application of Bruck Ryser Chwola theorem extensively. A test design constructed using the described method was found to be existing with an efficiency of 76.38%. Sum constructed designs are more applicable in agricultural fields as witness in case of KEFRI-Kenya.

Key words: Randomized Complete Block Design, Symmetric Balanced Incomplete Block Design, Automorphic Symmetric Balanced Incomplete Block Design

Introduction
Apart from Randomized Complete Block Designs (RCBDs), Balanced Incomplete Block Designs (BIBDs) are among the structures that are mostly studied in design theory under combinatorial mathematics. BIBDs came into existence through the efforts according to Yates, (1936). This author defined a BIBD as an arrangement of b blocks in a manner that exactly k distinct objects are contained in each block, with each object occurring in exactly r different blocks, and every pair of any distinct objects occurring together in exactly λ given blocks. According to Yates (1936), many authors, researchers and designers have paid attention to the construction of BIBD leading to several techniques available for the construction of BIBDs. This includes the trade-off method, symmetric repeated difference method, variety cutting and construction from finite permutation geometries among others. Bose (1938) noted that the Fisher Inequality for BIBD referred to the variables (v, b, r, k, λ) as the necessary parameters of any BIBD. Rao (1960) while studying on the duals of BIBDs revealed that BIBDs and their duals are variance balanced like the randomized complete block designs (RCBD). This author further added that the designs which are variance balanced are among the binary block designs whose block sizes are k less than number of the provided treatments. A method described as sum construction of automorphic symmetric BIBD has been presented in details.

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Balance Incomplete Block Designs

The generation, construction and analysis of BIBD is a standard combinatorial problem. Combinatorial mathematics in design theory is a branch of mathematics dealing with existence or non-existence, generation and properties of systems of finite sets having arrangements satisfying certain concepts like completeness, symmetry or balance. The concept of BIBD was initially developed in the field of design of agricultural experiments but later applied and expanded to other fields of science and art such as cryptography, coding theory, reliability and network among others. According to Fisher (1940), an Incomplete Block Design (IBD) is obtained by assigning less number of treatments k than the number of blocks b within each block in the design, that is k such that $b > k$. There are several types of IBD namely; Lattice Designs, BIBD, Partially Balanced Incomplete Block Designs (PBIBDs) and Youden Squares among others. A BIBD is among the most commonly used IBD where treatments are randomized within every block. In the experiments consisting of many blocks, it is not possible to assign all the treatments within each block. The goal as per Yates (1936) is to assign a sub-set of k ($k < b$) (the number of treatments in the experiment) treatments within each block. Using the argument that any pair of treatments λ within the design will occur: Therefore; pickings in every block of the design

$$\begin{pmatrix} v-2 \\ k-2 \end{pmatrix} \text{in} \begin{pmatrix} v \\ k \end{pmatrix}.$$

$$\lambda \frac{\begin{pmatrix} v \\ k \end{pmatrix}}{\begin{pmatrix} v-2 \\ k-2 \end{pmatrix}} = b$$

$$\lambda \frac{v!(k-2)!(v-k)!)}{k!(v-k)!(v-2)!} = b$$

$$\frac{\lambda v(v-1)}{k(k-1)} = b$$

$$rv = bk$$

Even if the given parameters satisfy the above conditions it is not always automatic that arranging the treatments in blocks will give rise to a corresponding design. Irrespective of several studies in the field of design theory, the necessary and sufficient conditions to be satisfied by the parameters for the existence of a BIBD are not settled very well. However, it has been shown that even if all of the non-existence or existence conditions are met it does not necessarily imply that a BIBD must always exist. For example, a BIBD does not

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exist for v = 15, k = 5, r = 7, λ = 2 and b = 21. Designs that have their parameters satisfying all conditions but still fail to exist are said to be non-orthogonal, however, such designs are believed to be balanced. That is to say they are not orthogonal because treatments are confounded with both blocks and plots within blocks. They are said to possess the balance since all comparisons between treatments and blocks are confounded to the same extent as with plots within blocks. It can be shown that for any BIBD the value of canonical efficiency factor, popularly known as proportion of the information given within the blocks is $e_2 = \frac{vk}{r\lambda}$ and the value of the canonical efficiency factor between the blocks is $e_1 = 1 - e_2$. These proportions are called the canonical efficiency factors. For a particular randomized term (treatments) the canonical efficiency factors e_1 and e_2 are always values ranging between 0 and 1.

A general definition of automorphism is given as a symmetry preserving the permutation or bijective function from a design onto itself in the study by Yasmin, Ahmed and Akhatar (2015). In the current study we provide an elaborative explanation on the construction of automorphic using a Group Theory approach and apply the concept to design analysis of experiment: In the context of group theory, an automorphism is considered to be an isomorphism from a Group to Itself.

Motivation

The involvement of mathematicians in solving problems dealing with the arrangement of a finite number of objects in sets or patterns that are known to be satisfying some given or established conditions, began in 1782 with the origin of BIBD dating back to 1936, when Yates (1936) introduced the analysis of BIBD using inter-block and intra-block analysis of information obtained from the earlier set experiments. Further contributions pertaining BIBDs were made by Bose and Nair (1938) in the late 1930s, concerning the structure and construction of BIBDs. Since then, the generation of block designs remains unresolved problem in combinatorial mathematics. Some methods of construction of BIBDs have been suggested, these includes the trade-off method, difference method, variety cutting and construction from finite permutation groups among others. However, the construction techniques have not been exhausted because there are many parameter sets for which the existence of BIBDs have not been determined. Due to many open questions and conjectures about existence of BIBDs which still remain unresolved, thus, this study has developed an algorithm for the sum construction of automorphic symmetric BIBDs alongside the efficiency of such designs have been determined. In the construction of such designs we employed mathematical combinatorial techniques while developing the algorithm in a Python 3.6.3 program.

Literature Review

This section introduces related studies, the methodology employed, the findings and critiques of the cited studies. Design and experiment as a subject was founded by a profound statistician (Fisher, 1940). This author gave the three principles on designs of experiments namely; randomization, replication and blocking. The BIBDs and PBIBDs were introduced by Yates (1936) in agricultural experiments. Basing on the three principles, Bose (1938) developed the construction of BIBD and its properties.

Consequently, several authors have discussed various properties of designs from various points of applications a fact that has led to a series of methods of construction of BIBDs as argued by Choi and Yi (2016). A gap of knowledge still exists on the sufficient or even necessary conditions for non-existence or existence of BIBDS a reason for which this study has provided an in depth review on the sum construction of automorphic symmetric BIBD basing on the build-up of the three principles of experimental design.

A computation of the A-efficiency of a design the efficiency table of comparison between A-efficient and A-optimal designs with respect to number of blocks. Further, the study findings advised that uniform conditions should be maintained while comparing a number of treatments so as to provide or make a precise measurement of treatment means. This ensures that the difference among treatment means remain so minimal and may only result from the application factors and not from some other extraneous factors. To achieve this, experimental trials are often grouped together into homogeneous blocks with conditions kept constant within such blocks. A concept that the current study employed in the collection of data set used in the analysis.

According to a study by Otulo et al. (2020) in which statistical procedures were used for combined independent test, a hypothesis for common mean vectors of two independent models or designs with different values of variance were tested. The efficiency of the designs involved in this study were computed on the basis of comparison of the mean sum of squares derived from the independent linear models. The method of testing different hypothesis proved suitable for certain conditions. These conditions included the equality of treatment effects and testing the significance of treatment variance parameter in BIBD. The findings revealed a significance in BIBD with bigger samples of block sizes. However, for smaller samples of block sizes the study did not achieve any notable significance under the problem of efficiency comparison for various designs remaining un-addressed. The current study purposes to improve the study by addressing both efficiencies of the Sum Constructed AUSBIBD. In order to eliminate heterogeneity and improve on the accuracy or efficiency of any BIBD, the current study has introduced a new concept of sum construction of AUSBIBD. In this newly constructed design heterogeneity is reduced to a greater extent than is possible with RCBD, LSD and initial SBIBD. As a further development in bridging the gap of knowledge along this line, the current study describes the efficiency of sum constructed AUSBIBD.

Kelechi (2012) on the construction of symmetric BIBD asserted that the block designs are many sub-sets with similar properties. These properties must satisfy some conditions which are important to certain application in the field of study of experimental design, software testing, algebraic geometry, and cryptography. This author widely captured the balanced incomplete block design (BIBD) concluding that when all the conditions pertaining to design are satisfied, then the symmetry remain un-interfered with. According to Otulo, Muga, Nyaare and Nyakinda, (2020) in their study on relative efficiency of sum constructed automorphic symmetric BIBDs, the incidence matrices of known symmetric BIBD and the sufficient conditions under which design becomes symmetric are key ingredients for the

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sum construction method of AUSBIBDs. The same authors concluded that the relative efficiency of a sum constructed automorphic symmetric balanced incomplete block designs are higher than the parent designs hence they reveal more information per block. They further indicated that pair-wise balanced designs are similar or almost similar to variance balanced designs. However, the designs needed a larger number of replications. The current study introduces the method of sum construction of developing SBIBD that are automorphic and equally provides insights on their efficiencies in details.

Methodology

Development of the sum construction method of AUSBIBD

Given two designs on an equal point set most of priority a symmetric balanced incomplete block design, the sum construction involves generating a collection of all the blocks in both of the arguably symmetric BIBDs used (Kelechi, A.C. (2012).). This study employs the idea of sum construction by supposing that if we have two automorphic symmetric BIBDs with parameters given as (v, k, λ_1) and (v, k, λ_2), respectively then the new design is obtained by adding a fixed value of λ to the treatments of the parent design at random. The theorems described below are the cornerstone of this present study:

Theorem 3.1: Sum Construction: Suppose BIBDS with (v, k, λ_1) and *a* (v, k, λ_2) as given parameters exists. Then a BIBD with (v, k, $\lambda_1 + \lambda_2$) as parameters exists. A simplified version of this theorem is given as follows:
If *a* (v, k, λ_1) - BIBD of design (X, A_1) and *a* (v, k, λ_2)-BIBD on design (X, A_2) exists on the given set X then a (v, k, ($\lambda_1+\lambda_2$)-BIBD exists on the set.

Corollary 3.2: If *a* (v, k, λ)-BIBD exists then *a* (v, k, $s\lambda$) – BIBD exists for all values of the integers s is greater than or equal to 1.
Note that the symmetric BIBDs produced by the above corollary with the value of $s \geq 2$ are always considered not to be simple designs, even if the initial (v, k, λ_1)-BIBD is simple. For $\lambda_1 > 1$, construction of simple BIBD is, in general, more difficult than construction of BIBD with repeated blocks (Lee, Kim, & Chung, 2004). The current study applies the sum construction technique to determine automorphic symmetric BIBD and a test to the efficiency of the sum constructed designs given priority.

Development of Algorithm on Sum Construction of AUSBIBD

Having picked on two designs on the same point set, all believed to be SBIBD, sum construction method is made possible by forming a collection of all the blocks in the new designs that are as a result of one to one and invertible mapping of the blocks of the original design. By fixing the parameter λ_2 new AUSBIBD are obtained through a Sum Construction Method. In this study, the concentration is on the efficiency of sum constructed AUSBIBD.

Results and Discussion

Sum Construction Method

Given any two designs whose point sets are the same, arguably symmetric BIBD, sum construction involves forming a collection of all the blocks in both designs as argued by Otulo, Muga, Nyaare and Nyakinda (2020). By fixing some parameters, a new design is obtained through a method of construction known to this study as sum construction. In this work, we have concentrated on the sum construction of automorphic symmetric balanced incomplete block design. The theorems below have been fully employed in this work:

Theorem 4.1: If there exist *a* (v, k, λ_1) BIBD and *a* (v, k, λ_2) BIBD then there exists a (v, k, $\lambda_1 + \lambda_2$) BIBD.

Corollary 4.2: Suppose we have an existing (v, k, λ)-BIBD then a BIBD whose parameters are (v, k sλ) is also known to be existing for all integers $s \geq 1$.

Suppose that we have *a* (v, k, λ_1,) existing BIBD and *a* (v, k, λ_2) existing BIBD on the set *X* then a new BIBD can be obtained by adding the λ_s for the parent designs, the method is referred to as sum construction.

Theorem 4.3: If *a* (v, k, λ_1) existing BIBD of design (X_1,A_1) and *a* (v, k, λ_2) existing BIBD of design (X_1,A_2) are known to be on a set *X*, then a BIBD whose parameters are (v, k, ($\lambda_1 + \lambda_2$)) is known to be existing on the provided set.

Proof: Let $A = A_1 \cup A_2$ be the multi-set union of the set A_1 and A_2 then A is a multi-set of non-empty subsets of X, clearly $|X|$ = v, furthermore since every block in A_1 contains k points and we have that every block in A_2 also contain k points. Then it follows that A too has k points. Let x_1,y∈ X be such that x≠y , then the given pair (x, y) is known to be contained in exactly λ_1 blocks in the set A_1 and the pair (x, y) is equally known to be contained in exactly λ_2 blocks in the set A_2 therefore the given pair (x, y) is impliedly known to be contained in exactly $\lambda_1 + \lambda_2$ blocks in the set **A**. This is proved true for any arbitrarily selected pair of distinct points of the values x, y ∈ X, hence the given (X, A) is BIBD whose parameters are (v, k, $\lambda_1 + \lambda_2$).

Corollary 4.4: If a given design (X, A) whose parameters are known to be (v, k, λ) is arguably a BIBD existing on a set *X*, then for every value of the positive integer $s \geq 1$, a BIBD(X, A^*) whose parameters are (v, k *s*λ) is believed to exist on *X*.

Proof: Let $s \geq 1$ be a positive integer, then set $A^* = A \cup A \cup A \ldots\ldots.. \cup A$. be the union of the multiset A with itself up to *s* times. By theorem 4.3, the design (X, A^*) becomes a $(v, k, \lambda + \lambda, \ldots\ldots.. + \lambda) = (v, k\, s\lambda) - \text{BIBD}$ where the sum construction of the SBIBD is carried on the multi set union *s* times.

Theorem 4.5: Given **A** to be an incidence matrix of a BIBD whose parameters are (v, k, λ), then $\mathbf{AA^T}$=(r−λ)*I* r λJ, with *J* being a *r* × *v* matrix all elements one and $\hat{J}$ being a *v* × *b* matrix of similar entries comprising of all ones. Further, if any matrix **A** is known to satisfy the non-existence or existence conditions provided by Fisher and Yates (1938), as $\lambda(v-1) = r(k-1)$ and bk = rv, when k<v, then the incidence of **A** satisfy the condition $\mathbf{AA^T}$=(r−λ)*I* r λJ.

Examples of Sum construction symmetric BIBDs

In a bid to describe the results for the described sum construction method, we provide an illustration of the results on sum construction by considering a single design whose

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parameters are: b = 22, r = 11, v = 12, k = 6 and by letting $v = (0,1,2,\ldots,10,\infty)$. We fix our $\lambda_{2s'}$ for the second design and use the sum construction technique to generate up new designs for three different scenarios λ = 2, 4 and 5 that is;

Case 1: where $\lambda_2 = 2$

The following are the resulting designs which are automorphic in nature:

(1, 3,4,5,9, ∞)	(0, 2, 6, 7, 8, 10)
(3, 5,6,7,0, ∞)	(2, 4, 8, 9, 10, 1)
(5, 7,8,9,2, ∞)	(4, 6, 10, 0, 1, 3)
(7, 9, 10, 0, 4, ∞)	(6, 8, 1, 2, 3, 5)
(9, 0,1,2,6, ∞)	(8, 10, 3, 4, 5, 7)
(0, 2,3,4,8, ∞)	(10, 1, 5, 6, 7, 9)
(2, 4, 5,6,10, ∞)	(1, 3, 7, 8, 9, 0)
(4, 6,7,8,1, ∞)	(3, 5, 9, 10, 0, 2)
(6, 8, 9,10,3, ∞)	(5, 7, 0, 1, 2, 4)
(8, 10,0,1,5, ∞)	(7, 9, 2, 3, 4, 6)
(10, 1,2,3,7, ∞)	(9, 0, 4, 5, 6, 8)
(1, 3,4,5,9, ∞)	(0, 2, 6, 7, 8, 10)

Case 2: where we consider our $\lambda_2 = 4$

The following are the resulting automorphic symmetric designs:

(1, 3, 4, 5, 9,∞)	(0, 2, 6, 7, 8, 10)
(5, 7, 8, 9, 2,∞)	(4, 6, 10, 0, 1, 3)
(9, 0, 1, 2, 6,∞)	(8, 10, 3, 4, 5, 7)
(2, 4, 5, 6, 10,∞)	(1, 3, 7, 8, 9, 0)
(6, 8, 9, 10, 3,∞)	(5, 7, 0, 1, 2, 4)
(10, 1, 2, 3, 7, ∞)	(9, 0, 4, 5, 6, 8)
(3, 5, 6, 7, 0,∞)	(2, 4, 8, 9, 10, 1)
(7, 9, 10, 0, 4, ∞)	(6, 8, 1, 2, 3, 5)

Case 3: where $\lambda_2 = 5$

The following are the results:

(1, 3, 4, 5, 9,∞)	(0, 2, 6, 7, 8, 10)
(6, 8, 9, 10, 3,∞)	(5, 7, 0, 1, 2, 4)
(0, 2, 3, 4, 8,∞)	(10, 1, 5, 6, 7, 9)
(5, 7, 8, 9, 2,∞)	(4, 6, 10, 0, 1, 3)
(10, 1, 2, 3, 7,∞)	(9, 0, 4, 5, 6, 8)

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(4, 6, 7, 8, 1,∞)
(9, 0, 1, 2, 6,∞)
(3, 5, 6, 7, 0,∞)
(3, 5, 9, 10, 0, 2)
(8, 10, 3, 4, 5, 7)
(2, 4, 8, 9, 10, 1)

The existence of the parent design was tested using Bruck-Ryser-Chowla Theorem according to Chowla and Ryser, (1950). The parameters of the design were replaced into the relationship describing the above theorem for existence test. A relevant example has been presented with the values of the parameters.

Efficiency of the designs

The efficiencies of the designs employed in this study were computed using the relationship as follows:

$$E = \frac{v}{v-1} \times \frac{k-1}{k}$$

Taking the test design with the values of the parameters specified above as b = 22, k = 6, v = 12 and λ = 2

The efficiency was computed

$$\frac{12}{12-1} \times \frac{6-1}{6} = \frac{60}{66}$$ Which is 90.91%

A test design (7, 3, 1) was equally used in determining the efficiency and a value 77.78% was obtained.

When other designs were subjected to computations of efficiencies the results obtained are populated as follows (Table 1). The main observation in this section of the study is that the efficiency increases with increase in the number of treatments in each block of the SBIBD. This is attributed by treatment ratio.

Table 1. Efficiency of the designs considered in the study

Design	Replication treatment ratio	Efficiency%
37,13,7	13:37	94.87
32,9,2	9:32	91.75
12,22,11,6,2	1:2	90.91
7,3,1	3:7	77.78

Conclusion

In this study various automorphic symmetric designs have been constructed using the sum construction method. It is worth concluding that automorphic symmetric BIBDS can be constructed from existing designs if the designs are symmetric in nature. This study further concludes that it is also possible to come up with new designs using automorphic symmetric BIBDS.

For the generation of any automorphic symmetric balanced incomplete block design using sum construction method, the following points are worth noting:

- List your design parameters in some fixed order.
- Identify the value that leads to a one to one mapping onto the design whose parameters are fixed as above

- Perform the sum construction of the automorphic SBIBD using theorem described in this study

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